9/27/21

14.1: Multivariable Functions

MATH-323

Space curve: $\vec{r}: I \to \mathbb{R}^n$

Def: A multivariable function (of real input and a real output) is a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$

function's name — domain — codomain

$\text{dom}(f) = D$ in this notation

$\text{ran}(f) = \{ f(\vec{x}) : x \in \text{dom}(f) \}$

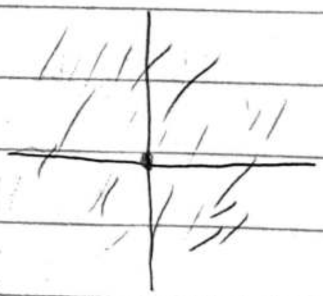
NB: If no domain is specified, we assume the biggest possible domain (i.e. the "natural domain")

Ex: $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

in this case, $\text{dom}(f) = \left\{ (x,y) \in \mathbb{R}^2 : \dfrac{x^2-y^2}{x^2+y^2} \text{ is defined} \right\}$

$= \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0 \}$

$= \{ (x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0) \}$

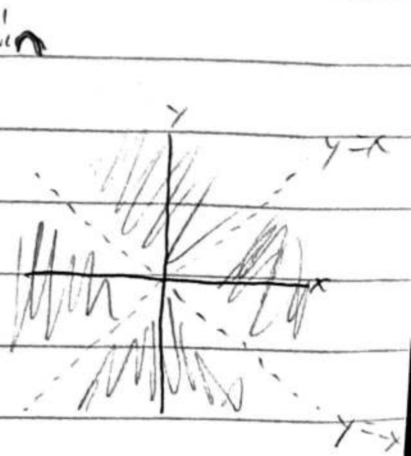


Ex: $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ has the same domain

Ex: $f(x,y) = \dfrac{x+y+1}{x^2-y^2}$

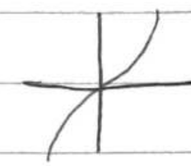
$\text{dom}(f) = \{ (x,y) \in \mathbb{R}^2 : x^2 - y^2 \neq 0 \}$

$= \{ (x,y) \in \mathbb{R}^2 : |x| \neq |y| \}$



Def: the graph of a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is
$\text{graph}(f) = \{(\vec{x}, f(\vec{x})) : \vec{x} \in \text{dom}(f)$

Ex (from Calc I) $f(x) = x^3$



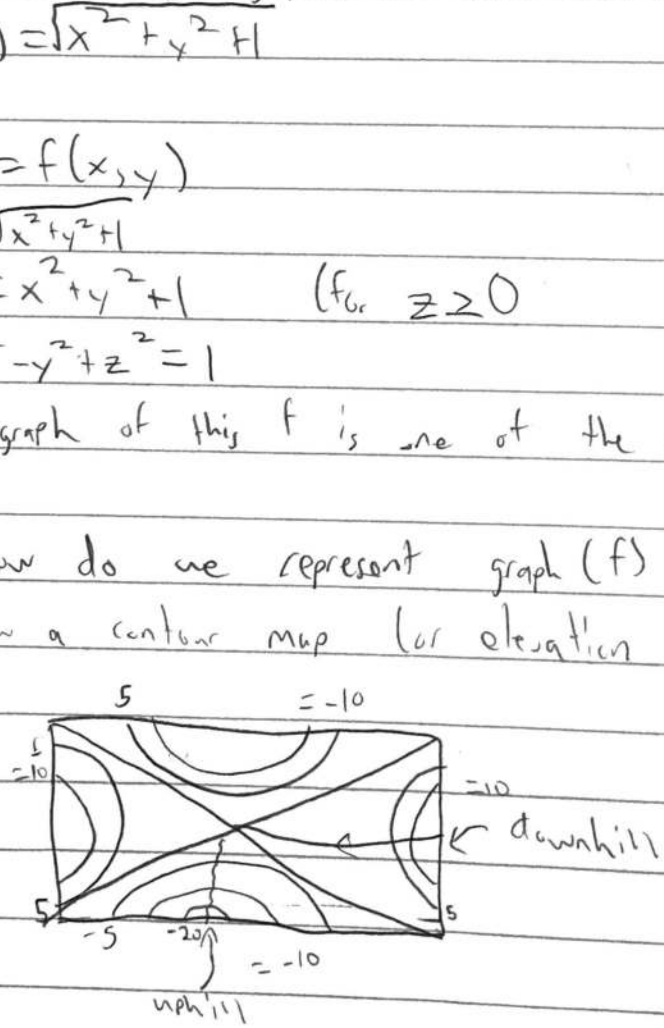
If $n=2$, this becomes
$\text{graph}(f) = \{(x,y), f(x,y)) : x \in \text{dom}(f)$
i.e. This is a picture of $z = f(x,y)$'s solution set

Ex: What does graph(f) look like for
$f(x,y) = \sqrt{x^2+y^2+1}$

Sol: $z = f(x,y)$
i.e. $z = \sqrt{x^2+y^2+1}$
i.e. $z^2 = x^2+y^2+1$ (for $z \geq 0$
i.e. $-x^2-y^2+z^2 = 1$
so the graph of this f is one of the sheets of this 2-sheet hyperboloid

Q: How do we represent graph (f) for a 2-variable function?
A: Draw a contour map (or elevation map or level curves)

Picture:



Ex: in 4-dimensions: The hypersphere

$S^3 = \{\vec{x} \in R^4 : |\vec{x}| = 1$

(x, y, z, w) $\qquad |w| \leq 1$

once $w = k$ is fixed

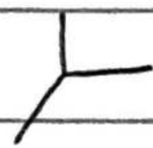
$$\sqrt{x^2 + y^2 + z^2 + k^2} = 1$$

$$x^2 + y^2 + z^2 = 1 - k^2$$

sphere of radius $\sqrt{1-k^2}$ about origin

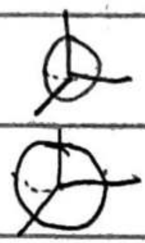
we get thus a movie describing the hypersphere:

("w = time")



$w = -1$

$w = -\frac{1}{2}$



small spheres

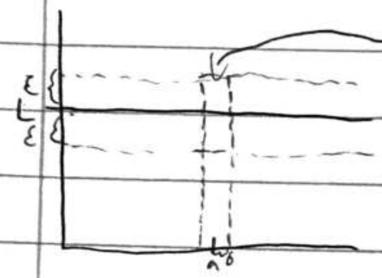
$w = 0$

bigger

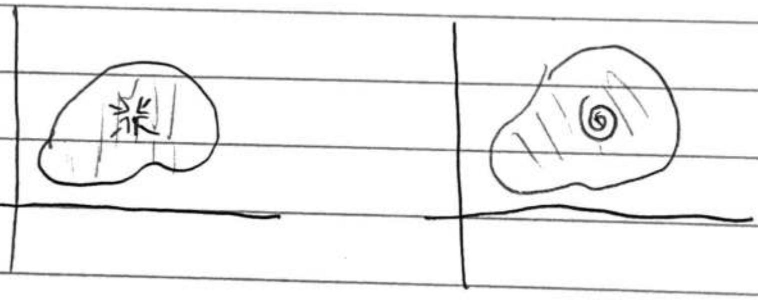
14.2: Limits and Continuity of Multivariable Functions

In Calc III, the formal definition of a limit goes like so:

Def: Let f be a multivariable function and let $\vec{a} \in \mathbb{R}^n$ be a limit point of the domain of f. The limit of f as $\vec{x}$ tends to $\vec{a}$ is $L \in \mathbb{R}$ when (for all unit vectors $\vec{u} \in \mathbb{R}^n$
For all $\varepsilon > 0$ there is a $\delta > 0$ for all $\vec{x} \in \text{dom}(f)$
we have $\|\vec{x} - \vec{a}\| < \delta$ $|f(\vec{x}) - L| < \varepsilon$



all of $f(x)$ lives in that rectangle near a



NB: This definition is hard to use... In practice, we'll want to use the following proposition in its place: (multivariable version of "one-sided" limits)

Prop: (Curves Criterium for Limits): suppose f is a multivariable function and $\vec{a}$ is a limit point of its domain
$\lim_{\vec{x} \to a} f(x) = L$ iff for all space curves $\vec{r}(t)$ in $\text{dom}(f)$ such that $\lim_{t \to \infty} \vec{r}(t) = \vec{a}$ we have $\lim_{t \to \infty} f(\vec{r}(t)) = L$

Notation: $\lim_{x \to a} f(\vec{x}) = L$ | **Alt:** $f(x) \to L$ as $\vec{x} \to \vec{a}$

Ex show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist

Sol: Consider the collection $L_{a,b}(t) = \langle at, bt \rangle$
where $(a,b) \neq (0,0)$ of
Observe $\lim_{t\to 0} \ell_{a,b}(t) = \langle 0,0 \rangle$

For $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, we know $f(\ell_{a,b}(t)) = f(at, bt)$

$$= \frac{at^2-bt^2}{at^2+bt^2}$$
$$= \frac{a^2-b^2}{a^2+b^2}$$

$\therefore$ if it exists we have
$\lim_{t\to 0} f(\ell_{a,b}(t)) = L$ for all a,b
$\lim_{t\to 0} f(\ell_{a,b}(t)) = \lim_{t\to 0} \frac{a^2-b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$

but if $a=1$, $b=0$ we would have $L=1$
and if $a=1=b$ we would have $L=0$